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Superradiant laser: First-order phase transition and non-stationary regime

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Abstract. We solve the superradiant laser model in two limiting cases. First the stationary low-pumping regime is considered where a first-order phase transition in the semiclassical solution occurs. This discontinuity is smeared out in the quantum regime. Second, we solve the model in the non-stationary regime where we find a temporally periodic solution. For a certain parameter range well-separated pulses may occur.

PACS. 42.50.-p Quantum optics – 42.50.Fx Cooperative phenomena; superradiance and superfluorescence – 42.50.Lc Quantum fluctuations, quantum noise, and quantum jumps

1 introduction

Superfluorescent pulses can be produced by N collectively radiating identical atoms [1–3] as these atoms decay from an initially excited state to the ground state. In contrast to such transient behavior would be the stationary output of the superradiant laser discussed in [4–6]. Collective behavior would be manifested in the proportionality of the output intensity to N^2 and of the linewidth to N^{-2}. Moreover, as was shown in [4,5] such a laser could display nearly perfect squeezing of the intensity fluctuations. The theory of a superradiant laser has up to now only been considered semiclassically in the stationary regime. As we shall show in the present paper, the semiclassical solution needs some quantum mechanical corrections for a finite number of atoms and low pumping amplitudes. We also propose to extend previous investigations towards a regime not allowing for a time independent stationary solution.

As in references [4,5] we consider the simplest model of a superradiant laser which accounts for N three-level atoms placed inside a resonator, Figure 1. We assume a resonant coherent two-photon pump process $0 \to 2$. The two transitions $2 \to 1$ and $1 \to 0$ are coupled to two resonant cavity modes a, b, which are assumed to be so strongly damped that they are kept in adiabatic slavery by the atoms.

The atoms are described by the collective population ($i = j$) and polarization ($i \neq j$) operators $S_{ij} = \sum_{\mu=1}^{N} S_{ij}^{\mu} = \sum_{\mu}(|i\rangle\langle j|)^{\mu}$. We shall represent these atomic observables by creation and annihilation operators $z_i^\dagger, z_i$

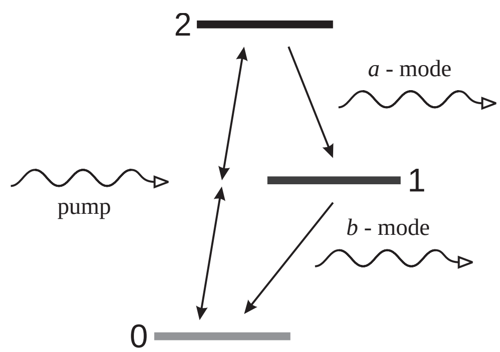

Fig. 1. Scheme of the 3-level superradiant laser.

with $S_{ij} = z_i^\dagger z_j$ and $[z_i, z_j^\dagger] = \delta_{ij}$; the two modes will be described by photon annihilation and creation operators $a, b, a^\dagger, b^\dagger$. The Hamiltonian H_0 for atoms and field modes reads

$$H_0 = -i\hbar g_{12}(a z_2^\dagger z_1 - a^\dagger z_1^\dagger z_2) + i\hbar g_{01}(b z_1^\dagger z_0 - b^\dagger z_0^\dagger z_1) + i\hbar \Omega (z_2^\dagger z_0 - z_0^\dagger z_2). \quad (1)$$

Here Ω is the amplitude of the external classical pump field, g_{12} and g_{01} are the atom-field coupling constants for the transition $2 \leftrightarrow 1$ and $1 \leftrightarrow 0$, respectively. As the modes are damped we have to add the two irreversible time rates of change for the mode amplitudes a, b

$$\left(\frac{\partial a}{\partial t}\right)_{irr} = -\kappa_a a(t) + \sqrt{2\kappa_a}\,\eta_a(t), \quad (2)$$

$$\left(\frac{\partial b}{\partial t}\right)_{irr} = -\kappa_b b(t) + \sqrt{2\kappa_b}\,\eta_b(t), \quad (3)$$

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where κ_a, κ_b are the two damping constants and $\eta_a(t), \eta_b(t)$ the corresponding quantum Langevin forces. The latter forces ensure the preservation of the Bose commutators $[a(t), a^\dagger(t)] = [b(t), b^\dagger(t)] = 1$. The Heisenberg-Langevin equations for the system with cavity damping are now

$$\dot{z}_1^\dagger = -g_{12} a z_2^\dagger + g_{01} b^\dagger z_0^\dagger, \tag{4}$$

$$\dot{z}_0^\dagger = -\Omega z_2^\dagger - g_{01} b z_1^\dagger, \tag{5}$$

$$\dot{z}_2^\dagger = \Omega z_0^\dagger + g_{12} a^\dagger z_1^\dagger, \tag{6}$$

$$\dot{a} = -g_{12} z_1^\dagger z_2 - \kappa_a a + \sqrt{2\kappa_a}\eta_a, \tag{7}$$

$$\dot{b} = -g_{01} z_0^\dagger z_1 - \kappa_b b + \sqrt{2\kappa_b}\eta_b. \tag{8}$$

These equations have been solved before in [5] under some assumptions that we shortly recapitulate here. First the b-mode is taken to be strongly damped so it can be adiabatically eliminated. For later use we define the two damping constants γ_a, γ_b:

$$\gamma_a = \frac{g_{12}^2}{\kappa_a}, \quad \gamma_b = \frac{g_{01}^2}{\kappa_b} \cdot \tag{9}$$

Then an analytic solution can be given in the semiclassical limit $N \gg 1$ in which the observables are represented as a sum $X = \bar{X} + \delta X$ of a dominant classical term $\bar{X} \sim N$ and a "small" operator valued fluctuation δX. The classical term is evaluated in the stationary regime by dropping $\dot{X}$ and the Langevin forces. The resulting solutions can be expressed in terms of a dimensionless coupling strength c and an effective pump strength p:

$$c = \frac{\gamma_a}{\gamma_b}, \quad p = \frac{\Omega}{N\sqrt{c}\gamma_b} \cdot \tag{10}$$

The solution for the level populations and the field amplitude is

$$\bar{S}_{00} = \frac{Nc(1-p)}{1+c}, \qquad \bar{S}_{11} = Np,$$
$$\bar{S}_{22} = \frac{N(1-p)}{1+c}, \qquad \bar{a} = \frac{N\gamma_b}{g_{12}} \frac{c\sqrt{p(1-p)}}{\sqrt{1+c}}, \tag{11}$$

which is stable under the constraints $0 \le p \le 1$ and $c > 1$ in the case we are interested in (additional adiabatic elimination of the a-mode). Notice that for $p \to 0$ and $p \to 1$ some of the observables vanish. This is in conflict to the assumption of a large mean value and small fluctuations, so the semiclassical approximation will break down at these points. Moreover, this solution cannot be correct in the limit $p \to 0$ since it does not correspond to the ground state $\langle S_{00} \rangle = N$. It is physically clear, however, that without pumping all atoms will eventually settle in the lowest state, due to damping of the levels 2 and 1. To improve on the semiclassical prediction of a discontinuity of $\langle S_{22} \rangle$ for $p = 0$ we have to consider the full quantum mechanical solution. This will be done in two ways. First, we develop a perturbative small-p expansion. Second, we give a recurrence relation for the whole stationary density operator ρ which can be evaluated numerically.

In the final section of this paper we shall take a look at the regime where (11) is not a stable solution anymore.

2 First-order phase transition in the stationary regime

For the discussion of the quantum mechanical solution we will switch to the master equation of the superradiant laser. Additionally we assume the a-mode also to be so strongly damped that we can eliminate it adiabatically. The master equation then reads

$$\dot{\rho} = (L_{02} + L_{21} + L_{10})\rho \tag{12}$$

with

$$L_{02}\rho = \Omega[S_{02} - S_{20}, \rho], \tag{13}$$

$$L_{21}\rho = \gamma_a\{[S_{12}, \rho S_{21}] + [S_{12}\rho, S_{21}]\}, \tag{14}$$

$$L_{10}\rho = \gamma_b\{[S_{01}, \rho S_{10}] + [S_{01}\rho, S_{10}]\}. \tag{15}$$

For the discussion we expand ρ in the fully symmetric and normalized states $|1^m; 2^l\rangle$ with l atoms in level 2, m atoms in level 1, and $N - m - l$ atoms in level 0. The short-hand notation $|0\rangle \equiv |1^0; 2^0\rangle$ is used for the ground state.

2.1 Second-order perturbation expansion

In this section we take a closer look at the neighborhood of the point $p = 0$. For zero pumping the atoms will eventually all settle in the ground state, so the stationary density operator is $\bar{\rho}(p = 0) = |0\rangle\langle 0|$. For small non-zero pumping $\bar{\rho}$ may be expanded in a series

$$\bar{\rho} = \rho^{(0)} + \rho^{(1)} + \rho^{(2)} + ... \tag{16}$$

where $\rho^{(n)} \propto p^n$. With $\lambda \equiv L_{21} + L_{10}$ the terms up to second order are

$$\rho^{(0)} = |0\rangle\langle 0|, \tag{17}$$

$$\rho^{(1)} = \lim_{t\to\infty} \int_0^t dt_1 e^{\lambda(t-t_1)} L_{02} e^{\lambda t_1} |0\rangle\langle 0|$$
$$= -\frac{pN^{3/2}}{\sqrt{c}} (|2^1\rangle\langle 0| + |0\rangle\langle 2^1|), \tag{18}$$

$$\rho^{(2)} = \lim_{t\to\infty} \int_0^t dt_1 e^{\lambda(t-t_1)} \int_0^{t_1} dt_2$$
$$\times L_{02} e^{\lambda(t_1-t_2)} L_{02} e^{\lambda t_2} |0\rangle\langle 0|$$
$$= \left(1 - p^2 N^2 \frac{N+c}{c}\right) |0\rangle\langle 0|$$
$$+ \frac{p^2 N^3}{c} |2^1\rangle\langle 2^1| + p^2 N^2 |1^1\rangle\langle 1^1|$$
$$+ \frac{p^2 N^3}{c} \sqrt{\frac{N-1}{2N}} \left(|2^2\rangle\langle 0| + |0\rangle\langle 2^2|\right). \tag{19}$$

$$0 = pN\sqrt{c}\Big\{\sqrt{(N-m-l+1)l}\bar{\rho}_{l-1,m,r} - \sqrt{(N-m-l)(l+1)}\bar{\rho}_{l+1,m,r} + \sqrt{(N-m-r+1)r}\bar{\rho}_{l,m,r-1} - \sqrt{(N-m-r)(r+1)}\bar{\rho}_{l,m,r+1}\Big\} + c\Big\{2m\sqrt{(r+1)(l+1)}\bar{\rho}_{l+1,m-1,r+1} - (m+1)(l+1)\bar{\rho}_{l,m,r} + 2(m+1)\sqrt{(N-m-l)(N-m-r)}\bar{\rho}_{l,m+1,r} - m(2N-2m+2-l-r)\bar{\rho}_{l,m,r}\Big\} \quad (24)$$

These entail the mean occupation numbers

$$\langle S_{00}\rangle = N - p^2N^2\frac{N+c}{c} + \mathcal{O}(p^3), \quad (20)$$

$$\langle S_{11}\rangle = p^2N^2 + \mathcal{O}(p^3), \quad (21)$$

$$\langle S_{22}\rangle = \frac{p^2N^3}{c} + \mathcal{O}(p^3). \quad (22)$$

Starting from the ground state the corrections are proportional to p^2 and $\langle S_{22}\rangle$ is of order N (as the semiclassical solution) for $p = 1/N$. Thus the discontinuity of the semiclassical solution is smeared out over a range $0 \le p \le 1/N$. Further calculations show that the fluctuations of the observables vanish for $p \to 0$. All of this is reminiscent of a first-order phase transition.

2.2 Recurrence relation

Having studied the perturbation expansion we now prepare to look for a complete quantum mechanical description. To that end we again expand the density operator ρ in the states $|1^m; 2^l\rangle$ introduced above. One may check that coherences with respect to different numbers of atoms in level 1 decay to zero. We take this into account by the simple ansatz

$$\bar{\rho} = \sum_{\substack{0\le m+r\le N\\ 0\le m+l\le N}} \bar{\rho}_{l,m,r}|1^m; 2^l\rangle\langle 1^m; 2^r|. \quad (23)$$

This expansion is inserted into (12) and we find a recurrence relation for the stationary solution $\bar{\rho}_{l,m,r}$

see equation (24) above.

We have not managed to find an analytical solution to this equation, but have solved it numerically. Since the number of involved variables is roughly proportional to N^3, only up to 30 atoms have been considered. The results are shown in Figure 2. Beside the corrections at $p = 0$ one can make out that the semiclassical solution has to be corrected for $p \to 1$ as well, as explained above. In the intermediate p range the semiclassical result is quite good, even for the moderate values of N studied. Since the transition from quantum to semiclassical behavior can already be seen in our calculations, it seems not necessary to go to higher N.

3 Non-stationary regime

In this section we consider the non-stationary solution of our system in the bad-cavity limit. As indicated in [6] we expect to find a temporally periodic behavior of the laser field. So we set $c < 1$, where the semiclassical stationary solution is not stable. Starting from the Heisenberg-Langevin equations (4–8), we assume strong, saturated driving between levels 0 and 2,

$$z_2 = \sqrt{N}\sin(\Omega t), \quad (25)$$

$$z_0 = \sqrt{N}\cos(\Omega t). \quad (26)$$

With the assumption of very fast relaxing cavity modes we can once more eliminate a and b adiabatically,

$$a(t) = -\frac{g_{12}\sqrt{N}}{\kappa_a}\sin(\Omega t)z_1^\dagger + \sqrt{\frac{2}{\kappa_a}}\eta_a(t), \quad (27)$$

$$b^\dagger(t) = -\frac{g_{01}\sqrt{N}}{\kappa_b}\sin(\Omega t)z_1^\dagger + \sqrt{\frac{2}{\kappa_b}}\eta_b^\dagger(t), \quad (28)$$

and find a linear equation for $z_1^\dagger$

$$\dot{z}_1^\dagger = \left[\frac{\Gamma_a - \Gamma_b}{2} - \frac{\Gamma_a + \Gamma_b}{2}\cos(2\Omega t)\right]z_1^\dagger + A(t), \quad (29)$$

$$\Gamma_a = \frac{g_{12}^2N}{\kappa_a}, \quad \Gamma_b = \frac{g_{01}^2N}{\kappa_b}, \quad (30)$$

$$A(t) = -\sqrt{2\Gamma_a}\sin(\Omega t)\eta_a(t) + \sqrt{2\Gamma_b}\cos(\Omega t)\eta_b^\dagger(t). \quad (31)$$

Now equation (29) is easily solved

$$z_1^\dagger(t) = e^{u(t)}z_1^\dagger(0) + e^{u(t)}\int_0^t e^{-u(t')}A(t')dt' \quad (32)$$

with

$$u(t) = \frac{\Gamma_a - \Gamma_b}{2}t - \frac{\Gamma_a + \Gamma_b}{4\Omega}\sin(2\Omega t). \quad (33)$$

Clearly we encounter a periodic behavior with period $1/2\Omega$. We use this as time scale and define the dimensionless parameters s, d, and τ

$$s = \frac{\Gamma_b + \Gamma_a}{2\Omega} = \frac{1+c}{2p\sqrt{c}}, \quad (34)$$

$$d = \frac{\Gamma_b - \Gamma_a}{2\Omega} = \frac{1-c}{2p\sqrt{c}}, \quad (35)$$

$$\tau = 2\Omega t. \quad (36)$$

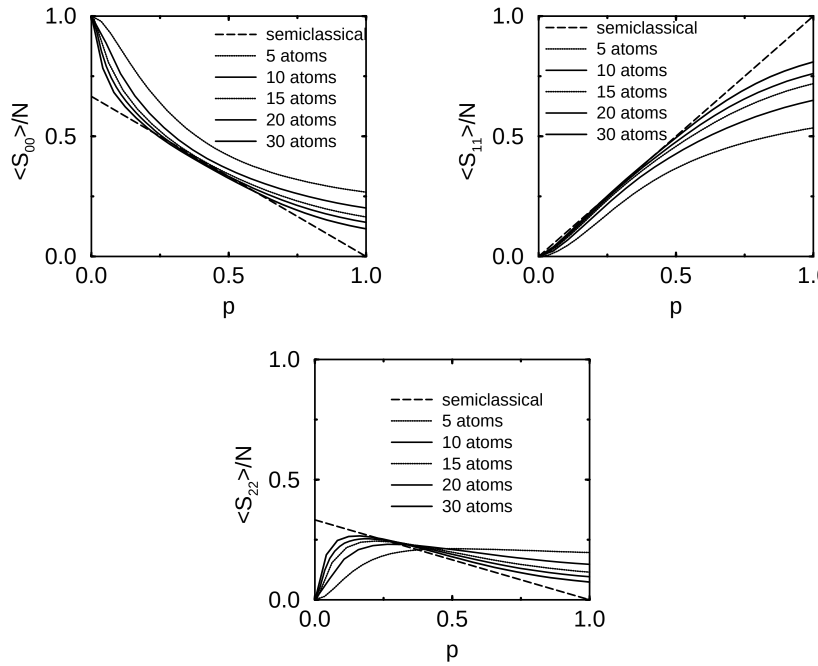

Fig. 2. Occupation numbers of the three levels derived through the recurrence relation. Even for our moderate number of atoms one can clearly see the transition from the quantum to the semiclassical regime. The discontinuity of $\langle S_{22} \rangle$ for $p \to 0$ in the semiclassical solution is smeared out over a range $1/N$.

From this the mean number of photons in the a-mode is

$$\left\langle a^\dagger(t)a(t)\right\rangle = \frac{\Gamma_a \Gamma_b}{\kappa_a \Omega} \sin^2(\tau/2) e^{-d\tau - s\sin(\tau)} \times \int_0^\tau \cos^2(\tau'/2) e^{d\tau' + s\sin(\tau')} d\tau' \equiv \frac{\Gamma_a \Gamma_b}{\kappa_a \Omega} \mathrm{Int}(\tau). \quad (37)$$

The time integration may be performed after expansion in terms of the Bessel functions $I_n(s)$,

$$e^{s\sin(\tau)} = \sum_{n=-\infty}^{+\infty} (-i)^n I_n(s) e^{in\tau}, \quad (38)$$

and after the death of initial transients $e^{-d\tau} \to 0$ we get

$$\mathrm{Int}(\tau) \to \frac{1}{2}\sin^2(\tau/2) e^{-s\sin(\tau)} \times \sum_{n=-\infty}^{+\infty} \frac{e^i n\tau}{d+in} (-i)^n I_n(s) \left(1 + i\frac{n}{s}\right). \quad (39)$$

To illustrate the temporal periodicity of this solution we give some examples for various parameters p, c in Figure 3. For the limiting case $s \gg 1$, which implies small p, we find well-separated pulses the width and height of which may be estimated in a Gaussian approximation. With the parameter $\epsilon = d/s$ we find

$$\mathrm{Int}(\tau) = \frac{1}{4}\frac{1}{s^2 p^2}\sqrt{2\pi p}\; e^{[2/p - d(\tau_{max} - \tau_{min})]} \times e^{-(\tau - \tau_{max})^2/2p} \quad (40)$$

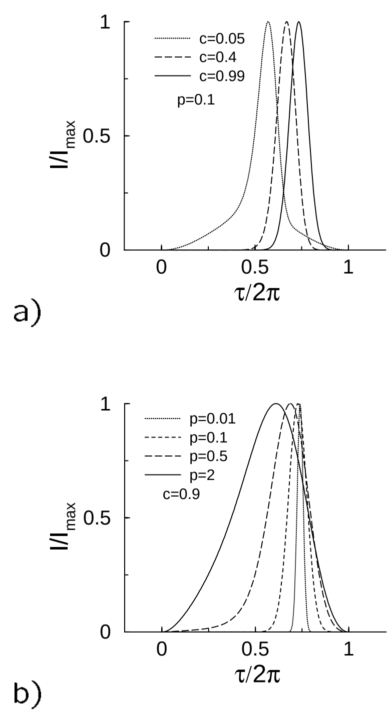

Fig. 3. The non-stationary pulsed solution for different parameters. As indicated by the calculations the pulse width is given by $\sqrt{p}$ and is independent of c.

with

$$\cos(\tau_{max}) = -\epsilon, \quad \sin(\tau_{max}) = -\sqrt{1-\epsilon^2}, \quad (41)$$

$$\cos(\tau_{min}) = -\epsilon, \quad \sin(\tau_{min}) = +\sqrt{1-\epsilon^2}. \quad (42)$$

$$S(\omega) \to \frac{\Gamma_a \Gamma_b}{16\Omega^3} \sum_n \sum_l \frac{(-1)^n I_n(s)\left(1+\frac{in}{s}\right)}{d+in} \left[I_{n+l+1}\left(\frac{s}{2}\right) - iI_{n+l}\left(\frac{s}{2}\right)\right]\left[I_{l+1}\left(\frac{s}{2}\right) + iI_l\left(\frac{s}{2}\right)\right] \times \left[\frac{1}{d+i(2n+2l+1+\omega/\Omega)} + \frac{1}{d+i(2n+2l+1-\omega/\Omega)}\right] \quad (45)$$

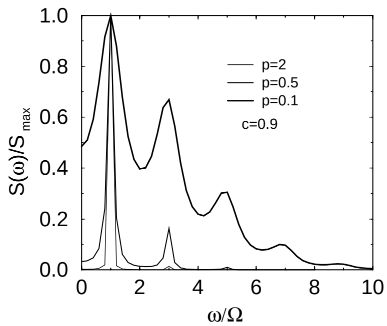

Fig. 4. Spectra of the pulsed solution for different parameters. As one expects the narrow pulse ($p = 0.1$) has higher harmonics than the broad pulse ($p = 2$).

So the pulse width is

$$\sigma = \sqrt{p} = \sqrt{\frac{\Omega\sqrt{\kappa_a \kappa_b}}{N g_{12} g_{01}}}\,. \quad (43)$$

Due to the assumptions of saturated driving and strongly damped cavity modes we have $\Omega \gg g_{12}, g_{01}$, $\kappa_a \gg g_{12}$, and $\kappa_b \gg g_{01}$. This entails the need for a large number of atoms to make σ small, *i.e.* to get narrow and well-separated pulses.

The time-dependent spectrum of a light signal which accounts for the width of the filter Γ is given as [7]:

$$S_\Gamma(\omega, t) = \int_0^t dt_1 \int_0^t dt_2 e^{-\Gamma(2t-t_1-t_2)} e^{i\omega(t_1-t_2)} \times \langle a^\dagger(t_1) a(t_2)\rangle. \quad (44)$$

Since we have explicit expressions for the $a, a^\dagger$ operators, we plug them in and after some integrations sending the width of the filter $\Gamma \to 0$ and the observation time $t \to \infty$ we get the following time-averaged expression

see equation (45) above.

We have used the same parameters p, c as in Figure 3 to illustrate the spectra of some of these pulses in Figure 4. Of course high harmonics of the Rabi frequency Ω are enhanced for $\sigma \to 0$.

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